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The development of magnetohydrodynamic flow due to an electric current discharge

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The development of the magnetohydrodynamic flow field due to the discharge of an electric current J_0 from a point on a plate bounding a semi-infinite viscous incompressible conducting fluid is considered. The flow field is the response of the fluid to the Lorentz force set up by the electric current and the associated magnetic field. The problem is formulated in terms of the dimensionless variable $(\nu t)^{\frac{1}{2}}/r$ and solved numerically. Here ν is the coefficient of kinematic viscosity, t the time from the application of the electric current and r the distance from the discharge. It is shown that the streamlines of the developing flow field in a cross-section through the axis of the discharge are closed loops about a stagnation point. As the flow field develops, the stagnation point moves to infinity along a ray emanating from the discharge with a speed proportional to $t^{-\frac{1}{2}}$. The steady state, within a distance r from the discharge, is practically established when $t = r^2/\nu$.

1. Introduction

The flow field set up by the discharge of an electric current in a semi-infinite incompressible fluid has applications in welding problems and has been considered by several authors (Lundquist 1969; Shercliff 1970; Sozou 1971; Sozou & English 1972). The current is discharged radially into the fluid from a point on a plane bounding the fluid. The flow field is set up by the rotational Lorentz force due to the current and the associated magnetic field. The work of Lundquist dealt with the linear problem (slow viscous flow). Shercliff retained the inertia terms in the momentum equation but ignored the fluid viscosity. This produced a singularity in the flow field everywhere along the axis of the discharge. This singularity is removed when the viscosity of the fluid is taken into consideration (Sozou 1971). The work of Sozou & English was more general, taking into account also the interaction of the velocity with the electromagnetic field. This interaction affects the overall velocity and current distribution when the parameter $4\pi\nu\sigma$, σ being the electrical conductivity of the fluid, is of order unity.

The solutions presented by these authors are based on similarity considerations. The flow field has a jet-like structure, similar to that of the momentum jet considered by Landau in 1944 (see Landau & Lifshitz 1959, p. 86) and by Squire (1951, 1952).

All the papers mentioned above study the steady state. The development of these flow fields has never been considered. The equations describing this development are very involved, even in the linear case. These problems have to be tackled numerically. Here we consider the development of the flow field due to a current discharged into a viscous liquid from a point on a fixed plate bounding the liquid. We formulate our problem under the assumption that, on switching on the discharge, the electromagnetic field is established instantaneously. We also assume that the effect of the velocity on the electromagnetic field is negligible, that is, we investigate the response of the fluid to the application of a steady Lorentz force. Our problem involves a fourth-order nonlinear partial differential equation, which we decompose into two second-order differential equations of mixed type. These we solve by iteration.

2. Formulation of the problem

We consider an incompressible viscous conducting fluid of density ρ and coefficient of kinematic viscosity ν occupying the semi-infinite region $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ of a spherical polar co-ordinate system (r, θ, ϕ). At $\theta = \frac{1}{2}\pi$ there is a fixed plate bounding the fluid and at the origin there is a current source (mathematically a point) which is suddenly switched on and supplies to the fluid region an electric current J_0. We assume that the electric current density $\mathbf{j}$ and magnetic induction $\mathbf{B}$ are set up instantaneously and the velocity field $\mathbf{v}$ is set up gradually as the response of the fluid to the rotational force $\mathbf{j} \times \mathbf{B}$. If we neglect the effect of the velocity on the electromagnetic field and assume that $\mathbf{j}$ is symmetric with respect to the axis $\theta = 0$ and purely radial, we find that (Sozou 1971) in e.m.u. $\mathbf{j}$ and $\mathbf{B}$ are given by

$$\mathbf{j} = \hat{\mathbf{r}} J_0/2\pi r^2, \quad \mathbf{B} = \hat{\boldsymbol{\phi}} \times 2J_0(1-\mu)/r(1-\mu^2)^{\frac{1}{2}}, \tag{1),(2}$$

where $\mu = \cos\theta$.

On eliminating the pressure from the momentum equation our governing equation becomes

$$\nabla \times [\partial \mathbf{v}/\partial t + (\nabla \times \mathbf{v}) \times \mathbf{v}] = -\nu \nabla \times \nabla \times \nabla \times \mathbf{v} + \nabla \times (\mathbf{j} \times \mathbf{B})/\rho. \tag{3}$$

The velocity field is obviously symmetric about the axis $\theta = 0$ and we satisfy the continuity equation by means of a stream function ψ. We assume

$$\psi = \nu r g(\mu, \lambda), \tag{4}$$

where $\lambda = (\nu t)^{\frac{1}{2}}/r$. $\mathbf{v}$ is then given by

$$\mathbf{v} = (-\nu/r)\,[g_\mu, (g - \lambda g_\lambda)/(1-\mu^2)^{\frac{1}{2}}, 0], \tag{5}$$

where a suffix denotes partial differentiation with respect to that variable. If we now use (1), (2) and (5), after a little algebra, (3) reduces to

$$(1-\mu^2) f_{\mu\mu} - 4\mu f_\mu + \lambda^2 f_{\lambda\lambda} + (4\lambda - 1/2\lambda) f_\lambda - 3 f g_\mu - g f_\mu + \lambda(f_\lambda g_\mu - f_\mu g_\lambda) = K/(1+\mu), \tag{6}$$

where $g(\mu, \lambda)$ satisfies the equation

$$g_{\mu\mu} + \lambda^2 g_{\lambda\lambda}/(1-\mu^2) = f(\mu, \lambda) \tag{7}$$

and $K = 2J_0^2/\pi\rho\nu^2$.

Equations (6) and (7) must be solved in the region $0 \leqslant \mu \leqslant 1$, $0 \leqslant \lambda \leqslant \infty$. It is expected that the steady state will be reached exponentially with respect to time. Thus, if we split the solution into a steady and an unsteady or λ-dependent component, for a fixed r as $t \to \infty$, that is, as $\lambda \to \infty$, the λ-dependent component of the solution will die out exponentially with respect to λ. This implies, of course, that as $\lambda \to \infty$

$$\lambda f_\lambda,\ \lambda^2 f_{\lambda\lambda},\ \lambda g_\lambda,\ \lambda^2 g_{\lambda\lambda} \to 0. \tag{8}$$

On eliminating f between (6) and (7), making use of (8), we find that in the steady state $g(\mu, \infty) = g_\infty(\mu)$ satisfies the equation

$$(1-\mu^2)\, g_\infty^{\text{iv}} - 4\mu g_\infty''' - 3g_\infty' g_\infty'' - g_\infty g_\infty''' = K/(1+\mu), \tag{9}$$

where a prime denotes differentiation. This equation was also obtained by Sozou [1971, equation (7)] for the corresponding steady-state problem. Thus, if we assume that (8) holds, as $t \to \infty$ our solution will have the expected behaviour. It is not very easy to solve (6) and (7) in the semi-infinite interval $0 \leqslant \lambda \leqslant \infty$ and the assumption of an exponential decay of the λ-dependent parts of f and g simplifies our computations considerably. This assumption, justified *a posteriori* by our results, enables us to restrict our solution to the strip $0 \leqslant \mu \leqslant 1$, $0 \leqslant \lambda \leqslant \Lambda$, where Λ is a finite quantity.

If we assume that g is specified, (6) is a linear equation in f which is elliptic in the region of interest and becomes parabolic on $\lambda = 0$ and on $\mu = 1$. Similarly, if we assume that f is specified, (7) is an elliptic equation in g which becomes parabolic on $\lambda = 0$ and on $\mu = 1$. We can specify g and solve (6) for f. That solution can be used in (7) to construct a better approximation to g, which is then used in (6) for a better approximation to f and so on.

The boundary conditions on f and g for the solution of (6) and (7) are

$$f(\mu, 0) = g(\mu, 0) = 0, \quad g(0, \lambda) = g_\mu(0, \lambda) = 0, \tag{10), (11}$$

$$g(\mu, \Lambda) = g_\infty(\mu), \quad f(\mu, \Lambda) = g_\infty''(\mu), \quad g(1, \lambda) = 0, \tag{12)–(14}$$

$$-4f_\mu + \lambda^2 f_{\lambda\lambda} + (4\lambda - 1/2\lambda) f_\lambda - 3fg_\mu + \lambda f_\lambda g_\mu = \tfrac{1}{2}K \quad \text{on} \quad \mu = 1. \tag{15}$$

Equation (10) means that when we switch on the electromagnetic field f and g are zero and (11) means that $\mathbf{v} = 0$ on the fixed plate. Equations (12) and (13) imply approximation of $f(\mu, \infty)$ by $f(\mu, \Lambda)$ and of $g(\mu, \infty)$ by $g(\mu, \Lambda)$. Equation (14) implies that $\mathbf{v}$ is finite on the axis $\mu = 1$. Equation (15) is derived from (6) under the assumption that $f_{\mu\mu}$ is finite on $\mu = 1$. In the special case $\lambda = \infty$, $f_{\mu\mu}$ is finite on $\mu = 1$ and for consistency $f_{\mu\mu}$ must be finite on $\mu = 1$ for all λ. Besides this we cannot see any reason why $f_{\mu\mu}$ should not be finite for all λ on $\mu = 1$. It must be noted that on $\mu = 0$ there is no explicit boundary condition for f, and g seems to be over-specified, since on $\mu = 0$ we specify g and its normal derivative. This is overcome by satisfying the condition $g(0, \lambda) = 0$ and then choosing $f(0, \lambda)$ such that $g_\mu(0, \lambda) = 0$. In view of (7) it is obvious that we must set

$$f(0, \lambda) = g_{\mu\mu}(0, \lambda). \tag{16}$$

This equation represents the boundary condition for f on $\mu = 0$.

3. Initial motions

For $\lambda \ll 1$ the dominant term on the right-hand side of (6) is $-f_\lambda/2\lambda$, except near $\mu = 0$, where owing to boundary-layer effects $f_{\mu\mu}$ need not be small. Thus, except very near $\mu = 0$, for small λ the solution of (6) may be approximated by

$$f \simeq -K\lambda^2/(1+\mu). \tag{17}$$

The appropriate solution of (7) is then found to be

$$g = \tfrac{1}{2}K\lambda^2\mu(1-\mu) \tag{18}$$

and thus from (4) we obtain

$$\psi = \frac{J_0^2}{\pi\rho}\frac{t}{r}\mu(1-\mu) \quad \text{for} \quad \nu t/r^2 \ll 1. \tag{19}$$

Equation (18) does not satisfy the condition $g_\mu(0,\lambda) = 0$, since the form for f given by (17) is not valid near $\mu = 0$. The approximation to ψ given by (19) is independent of ν. This expression corresponds to the solution of the linearized inviscid form of (3), which is valid initially, when the velocity is small, but only at distances to which the vorticity generated near the solid boundary has not had time to diffuse. For an investigation of the initial development of the flow near $\mu = 0$ we should construct a solution of (6) and (7) valid in an 'inner' region near $\mu = 0$ and match this with (17) and (18) in an intermediate region. This we do not propose to do but shall present a numerical solution of (6) and (7) under the boundary conditions (10)–(15). We note briefly, however, that we must have $f(0,\lambda) > 0$ and thus, for small λ, f changes sign as μ increases from zero to one. The condition $f(0,\lambda) > 0$ follows from (11), (16) and the fact that in the steady state ($\lambda = \infty$) $g_{\mu\mu} > 0$ on $\mu = 0$. $f(0,\lambda)$ must be of the same sign for all λ since if this were not the case, in view of (11) and (16), there would be a flow reversal near $\mu = 0$ at some time. This we consider unlikely.

4. The numerical method

Our fundamental equations (6) and (7) are of mixed type: they are elliptic within the domain of interest, $0 < \mu < 1$, $0 < \lambda < \Lambda$, and become parabolic on part of the boundary, that is, on $\mu = 1$ and on $\lambda = 0$. The numerical solution of (6) is particularly complicated since, as a consequence of the boundary condition (15), the line $\mu = 1$ is included in the solution domain for f. On the remaining part of the boundary, f may be determined from g by making use of (7), (10)–(12) and (14). We found it convenient to use the transformation (Bitsadze 1964, p. 58)

$$\eta = (1-\mu)^{\frac{1}{2}} \tag{20}$$

and hence express (6) in the form

$$(2-\eta^2)F_{\eta\eta} - (2\lambda G_\lambda - 2G - 6 + 7\eta^2)F_\eta/\eta + 4\lambda^2 F_{\lambda\lambda} + 2(8\lambda - \lambda^{-1} + \lambda G_\eta/\eta)F_\lambda + 6G_\eta F/\eta = 4K/(2-\eta^2), \tag{21}$$

where $F(\eta,\lambda) = f(1-\eta^2,\lambda)$ and $G(\eta,\lambda) = g(1-\eta^2,\lambda)$. Equation (21) is now elliptic throughout the region of interest $0 \leqslant \eta < 1$, $0 < \lambda < \Lambda$ and since f_μ and g_μ are finite everywhere, in the limit $\eta \to 0$ we must have

$$F_\eta(\eta,\lambda)/\eta = F_{\eta\eta}, \quad G_\eta(\eta,\lambda)/\eta = G_{\eta\eta}. \tag{22}$$

The boundary conditions for the solution of (21), obtained by transforming those for the solution for f, are

$$F(\eta, 0) = 0, \quad F(\eta, \Lambda) = g''_{\infty}(\mu), \tag{23),(24}$$

$$F_{\eta}(0, \lambda) = 0, \quad F(1, \lambda) = g_{\mu\mu}(0, \lambda). \tag{25),(26}$$

Equations (22) and (25) replace (15) and represent the condition that the partial derivatives of f are finite on $\mu = 1$.

Equation (7) is elliptic in the region of interest and g is specified on all boundaries. The condition $g_{\mu}(0, \lambda) = 0$ may be satisfied by assuming

$$g(\delta\mu, \lambda) = \tfrac{1}{2}(\delta\mu)^2 F(1, \lambda) \tag{27}$$

and using this as a boundary condition; that is we can solve (7) in the region $\delta\mu < \mu < 1, 0 < \lambda < \Lambda$ and assume that, on $\delta\mu$, g is given by (27). It is, of course, possible to transform (7) into an η, λ plane. In view of the relative simplicity of (7) and the fact that (7) is completely elliptic within the region of interest we did not pursue this approach.

For a given K we constructed the solution of our problem by iteration as follows. We specified an initial approximation for g, estimated the boundary values for F and then solved (21) for F. The constructed value of F was substituted in (7), which was solved for g. We then respecified $F(1, \lambda)$, using (26), and substituted the new g in (21), which was solved for a better approximation to F and so on. We repeated this process until convergence was obtained. We assumed that convergence was achieved when mesh-point values for F and g changed by less than 0·1 % between two successive iterations. Below we outline the numerical methods employed to solve (7) and (21).

The region $0 \leqslant \eta \leqslant 1$, $0 \leqslant \lambda \leqslant \Lambda$ was overlaid with a rectangular mesh with intervals $\delta\eta$ and $\delta\lambda$. The choice of a constant mesh length $\delta\eta$ implies that the corresponding mesh length $\delta\mu$ in the μ, λ plane is variable. This introduces little extra complication in the finite-difference representation of (7) since, using Taylor series, it is a straightforward matter to derive a second-order finite-difference formula for $g_{\mu\mu}$ using three unequally spaced points. The term $g_{\lambda\lambda}$ was approximated by the standard second-order central-difference formula. Thus (7) was approximated by a set of linear algebraic equations for g at the mesh points.

The difference representation chosen for (21) was rather more involved since this equation contains terms in F_{η} and F_{λ} whose coefficients can become large compared with the coefficients of $F_{\eta\eta}$ and $F_{\lambda\lambda}$, respectively. A method for dealing with such terms has been suggested by Allen (1962). Following his approach we set

$$(2-\eta^2) F_{\eta\eta} - (2\lambda G_{\lambda} - 2G - 6 + 7\eta^2) F_{\eta}/\eta = \alpha, \tag{28}$$

so that

$$4\lambda^2 F_{\lambda\lambda} + 2(8\lambda - \lambda^{-1} + \lambda G_{\eta}/\eta) F_{\lambda} + 6G_{\eta} F/\eta - 4K/(2-\eta^2) = -\alpha. \tag{29}$$

The basis of Allen's method is to solve the differential equations (28) and (29) as if they were ordinary, treating the coefficients as locally constant. Thus, for example, in the neighbourhood of the mesh point (η_i, λ_j) the solution of (28) may be written as

$$F = A + B \exp(-a_{ij}\eta) + \alpha\eta/b_{ij}, \tag{30}$$

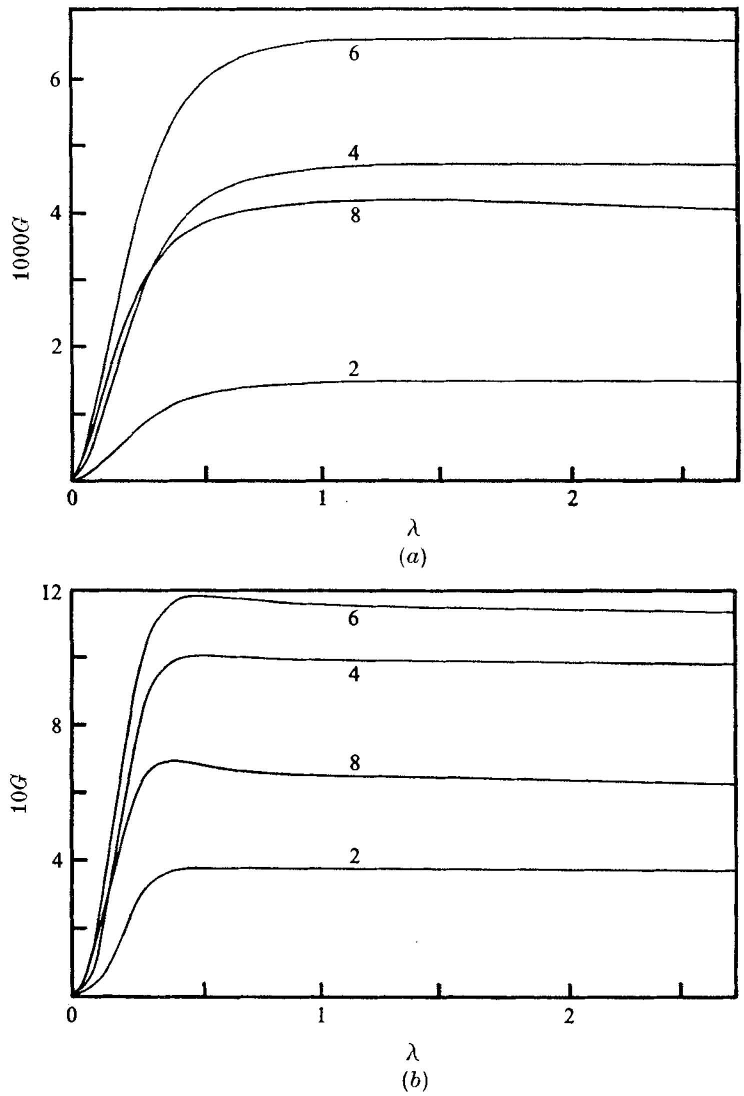

FIGURE 1. Values of G, as functions of λ, for specific values of η, for the cases (a) $K = 1$ and (b) $K = 150$. The numbers on the curves are values of 10η.

where A and B are arbitrary constants, $a_{ij} = -2[(\lambda G_\lambda - G - 3 + \frac{7}{2}\eta^2)/\eta(2-\eta^2)]_{ij}$ and $b_{ij} = (2-\eta_i^2)\,a_{ij}$. A finite-difference approximation to (28) may then be generated by eliminating A and B from the expressions for F_{ij}, F_{i+1j} and F_{i-1j}, obtained from (30). Equation (29) may be treated in a manner similar to that for (28), and by eliminating α from the resulting difference approximations to these equations we generate the finite-difference approximation to (21). A practical difficulty, however, is that the exponential terms associated with the local solution of (29) become very large for small λ. In order to avoid this we approximated (29) everywhere by using the standard second-order difference formula for $F_{\lambda\lambda}$ and a two-point backward difference formula for F_λ. The evaluation of $F(1, \lambda)$ from (26) was also based on second-order differences.

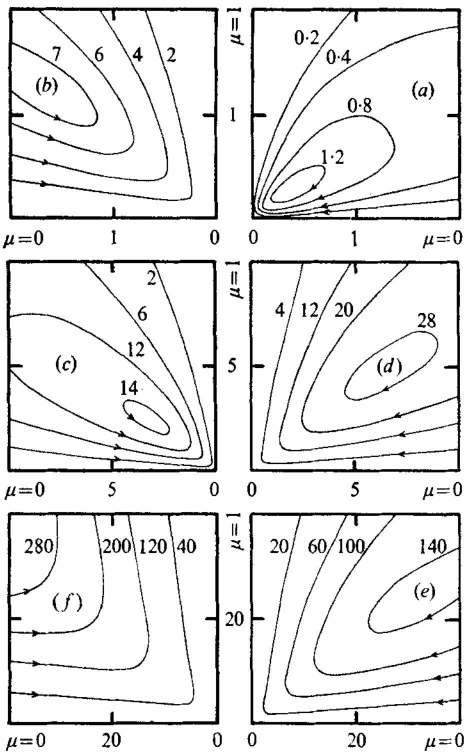

FIGURE 2

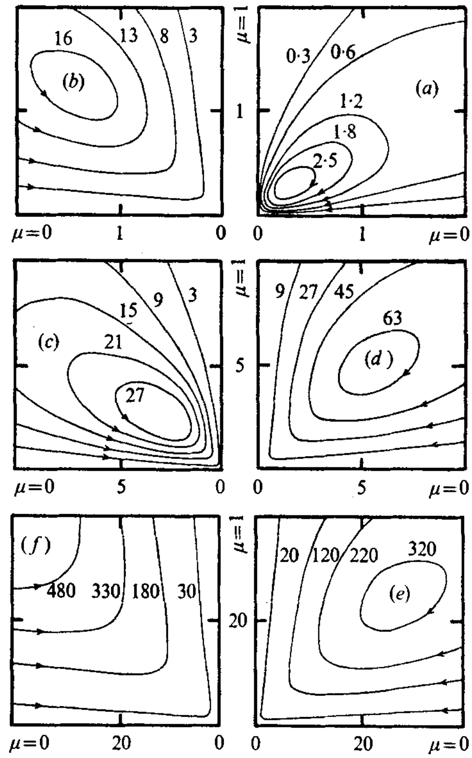

FIGURE 3

FIGURE 2. Streamlines of the developing velocity field for the case $K = 1$ at (*a*) $T \equiv (\nu t)^{\frac{1}{2}}/L = 0{\cdot}1$, (*b*) $T = 0{\cdot}5$, (*c*) $T = 1$, (*d*) $T = 2$, (*e*) $T = 10$ and (*f*) $T = \infty$. The numbers on the curves are values of $1000\psi/\nu L$, where L is a characteristic length. The distances along the axes are in units of L.

FIGURE 3. Streamlines of the developing velocity field for the case $K = 150$ at (*a*) $T = 0{\cdot}1$, (*b*) $T = 0{\cdot}5$, (*c*) $T = 1$, (*d*) $T = 2$, (*e*) $T = 10$ and (*f*) $T = \infty$. The numbers on the curves are values of $10\psi/\nu L$, where L is a characteristic length. The distances along the axes are in units of L.

The presence of the term F_λ/λ in (21) and the boundary conditions for F and g on $\lambda = 0$ implies that F must be evaluated very accurately near $\lambda = 0$, in order that the finite-difference scheme should be a good approximation to (21) near $\lambda = 0$. In order to obtain the correct behaviour for F near $\lambda = 0$ and improve the rate of convergence of the solution we employed (17) as a boundary condition on $\delta\lambda$ for $\mu \geqslant 0{\cdot}2$. For $\mu < 0{\cdot}2$ we made use of the finite-difference representation of (21) in order to accommodate the expected change in the sign of F. For small λ, values of g determined from the numerical solution of (7) agreed well with those predicted by (18).

The solutions of the difference equations representing (7) and (21) were obtained by successive over-relaxation. In the computations we set $\Lambda = \frac{8}{3}$ and chose the step lengths $\delta\lambda$ and $\delta\eta$ as $0{\cdot}033$ and $0{\cdot}05$, respectively.

5. Results and discussion

The steady-state flow field develops singularities – that is, it breaks down (Sozou 1971) – when the parameter K exceeds 300·1. Thus the developing flow field will exist for all λ provided that $K < 300{\cdot}1$. We performed computations for some K in the range 1–150. When K is $O(1)$

$$|f| \ll 1, \quad g \ll 1, \tag{31}$$

and (21) is, in effect, linear. When K is $O(100)$ the inequalities (31) are no longer valid everywhere and our problem is nonlinear.

Our results show that, for a given η, G increases with λ until it reaches a maximum and thence as λ increases G decreases monotonically to its steady value. For a given η, as K increases the maximum value of G, relative to its steady value, increases and also occurs at a lower value of λ. This is illustrated clearly in figure 1, where values of G are shown as functions of λ, for some specific η, for the cases $K = 1$ and $K = 150$. Inspection of figure 1 shows also that the value of G at $\lambda = 1$ is hardly different from that corresponding to the steady state. Detailed examination of the computer output reveals that $G(\eta, 1)$ is larger than $G(\eta, \infty)$ by about 2 % except very close to the axis of the current ($\eta = 0$), where the difference is of the order of 8 %. This implies that, within a hemisphere of radius r about the origin, the steady state is practically reached within a time r^2/ν. It also justifies our setting the time $\lambda = \infty$ at $\lambda = \Lambda = \frac{8}{3}$.

It is easy to deduce from figure 1 that, during its development, the flow field contains one stagnation point, in effect one stagnation ring about the axis $\eta = 0$. We note that $\psi = \nu(\nu t)^{\frac{1}{2}} G/\lambda$. Since as $\lambda \to 0$, $G/\lambda \to 0$, it becomes obvious from figure 1 that for a given value of η there exists a λ, say λ_0, for which the function G/λ is a maximum. Since $G(0, \lambda) = G(1, \lambda) = 0$, it is evident that there exists a λ_0, say λ_M, corresponding to a particular η, say η_M, such that $G(\eta_M, \lambda_M)/\lambda_M$ is a maximum. The point (λ_M, η_M) corresponds to a maximum in ψ and is a stagnation point.

In the physical, r, μ plane the streamlines, at any time t after the application of the current, are closed loops about the stagnation point $((\nu t)^{\frac{1}{2}}/\lambda_M, 1-\eta_M^2)$. As t increases the velocity field develops and the stagnation point moves to infinity along the line $\mu = 1-\eta_M^2$ with speed $\nu^{\frac{1}{2}}/2\lambda_M t^{\frac{1}{2}}$. Figures 2 and 3 show flow-field lines at various times for the cases $K = 1$ and $K = 150$. For $K = 1$, $\lambda_M \simeq 0{\cdot}24$ and $\mu_M \simeq 0{\cdot}58$ and for $K = 150$, $\lambda_M \simeq 0{\cdot}27$ and $\mu_M \simeq 0{\cdot}56$. From this and other computational results we examined, it is evident that as K increases so does λ_M. Thus the larger K is the longer it takes the stagnation point to propagate to infinity and the velocity field to develop. This is also evident from figures 2 and 3. Inspection of these figures shows that for a particular value of $T = (\nu t)^{\frac{1}{2}}/L$, where L is a characteristic length, say $T = 10$, the centre of the streamlines (maximum value of ψ) is further from the origin for the case $K = 1$ than for the case $K = 150$.

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